

An approach to Yang Mills perturbative and non-perturbative beta functions

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Using the global properties of the Yang Mills partition function we determine an all order perturbative beta function in the background gauge field method to find out that it has exactly the form of the beta function proposed by Pica and Sannino by analogy with the supersymmetric NSVZ beta function. We further compute the non-perturbative beta functions for the coupling constant and the theta angle in the background of an instanton field with winding number n . We solve for the theta angle in the non-perturbative region to determine that it is approximately zero. By extrapolating to the full QCD beta functions our result may constitute a solution to the strong CP problem.

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I. INTRODUCTION

The behavior of quantum chromodynamics is of utmost interest both at high energies where the coupling constant is small and perturbation theory makes sense and at low energies where the coupling constant is large and quarks form bound states, the hadrons. The perturbative beta function in QCD has been computed up to the fourth order in the minimal regularization scheme [1],[2]. In the background gauge field method beta function has been calculated at two loops in [3]-[4]. It is well known that the first two orders coefficients of the beta function are renormalization scheme independent and as such there exist a renormalization scheme where the beta function stops at the first two orders [5], [6], [7]. However it is argued that such scheme may pose difficulties with computing other quantities in the renormalized perturbation theory. It is quite generic that if only the first two coefficients of the beta function are renormalization scheme independent then one might obtain for higher orders at least in principle any well behaved function. However knowing the exact beta function is as important as the scheme that leads to it because one needs to approach all phenomena in the same context. In [8] Pica and Sannino proposed an all order beta function for the the QCD coupling constant inspired by the famous NSVZ $N = 1$ supersymmetric beta function [9]. The Yang Mills version of this has a simple formula as:

$$\beta_{YM} = -\frac{11}{3} \frac{\alpha^2}{2\pi} \frac{C_2(G)}{1 - \frac{\alpha}{2\pi} \frac{17}{11} C_2(G)}, \quad (1)$$

where $\alpha = \frac{g^2}{16\pi^2}$ and $C_2(G) = N$ where N is the number of colors.

Regular Yang Mills and QCD are by far more complicated models than their supersymmetric counterparts that have been discussed and elucidated in a series of groundbreaking works in [10]-[12]. In [12] Seiberg showed that at least for the $N = 2$ supersymmetric gauge theories the perturbative beta function stops at one loop and also introduced the first order non-perturbative effects for both the coupling constant and the theta angle. Of topical interest is not only the beta function for the coupling constant but also the behavior of the theta angle with the scale. Although early attempts have been made to extract at least the contribution of the theta angle to the coupling constant beta function [13], or the form of both beta functions with the scale [14] not definite answer has emerged in this direction. But the exact behavior of the the theta angle with the scale and the strong coupling constant is crucial as it might lead to insights with regard to the strong CP problem.

In this work we shall consider Yang Mills theory in the background gauge field theory and based on the properties of the partition function we shall derive the exact form of the complete beta function in the perturbative approach to determine that its expression is that suggested by Pica and Sannino in [8]. Moreover we shall also determine non-perturbative effect in the background of an arbitrary instanton field. In this context we estimate also the beta function for the theta angle. Our results have some common features with the Seiberg non-perturbative supersymmetric beta function and to other approaches but also significant differences. Based on the two complete beta function for the coupling constant and the theta angle we extract the behavior of the theta angle with the coupling constant at low energies where the coupling constant is large. We find that in this regime the theta angles runs to the effective value

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a zero. This results is in striking agreement with the experimental results like that for electric dipole momentum for the neutrons [15] that suggest that the theta angle is close to zero so it might constitute a solution to the strong CP problem. Of course in order to definitely claim so one might need to reconsider the same approach for full QCD with fermions. However our derivations here have the salient feature that one can include fermions without modifying the main aspects or altering the main results. A discussion in this sense will be made in the Conclusions section.

II. THE SET-UP

We start from the Yang Mills Lagrangian in the background gauge field method [16]:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4g^2} [F_{\mu\nu}^a + D_\mu A_\nu^a - D_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c]^2 - \\ & -\frac{1}{2g^2} (D^\mu A_\mu^a)^2 - \bar{c}^a (D^2)^{ac} c^c - \bar{c}^a D^\mu f^{abc} A_\mu^b c^c, \end{aligned} \quad (2)$$

where,

$$A_\mu^a \rightarrow B_\mu^a + A_\mu^a \quad (3)$$

and B_μ^a is the background gauge field and A_μ^a in the decomposition is the quantum gauge field fluctuation. Also,

$$(D_\mu)^{ac} = \partial_\mu \delta^{ac} + B_\mu^b f^{abc}, \quad (4)$$

where D_μ is the covariant derivative in the background gauge field method that it is applied to the quantum gauge field and to the ghosts.

Next we shall consider the partition function associated with the Yang Mills field in the background gauge field method:

$$Z = \int dA_\mu^a d\bar{c}^b dc^b \exp[i \int d^4x \mathcal{L}(B, A, c)]. \quad (5)$$

We allow for the possibility of a theta angle term and extend this partition function to:

$$Z_\theta = \int dA_\mu^a d\bar{c}^b dc^b \exp[i \int d^4x \mathcal{L}(B, A, c) + i\theta \int d^4x \mathcal{L}_1(B, A, c)] \quad (6)$$

where,

$$\mathcal{L}_1 = \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} [F_{\mu\nu}^a + D_\mu A_\nu^a - D_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c] [F_{\rho\sigma}^a + D_\rho A_\sigma^a - D_\sigma A_\rho^a + f^{abc} A_\rho^b A_\sigma^c]. \quad (7)$$

The calculation in the background gauge field are made in the saddle approximation where the linear term is set to zero. Since this is equivalent to asking that the background gauge field satisfies the equation of motion we shall consider this as a constraint that applies to all the calculations in the present work.

One can integrate the partition function to obtain the effective potential whose form is known in the first orders and envisioned in all orders:

$$\begin{aligned} Z_{\theta,B} = & \exp[-\Gamma_{eff}] = \\ & \exp[- \int d^4x \frac{1}{4g^2(\mu)} F_{\mu\nu}^a F^{a\mu\nu} + i\theta(\mu) \frac{1}{32\pi^2} \int d^4x \tilde{F}^{a\mu\nu} F_{\mu\nu}^a + \text{higher order gauge invariants}], \end{aligned} \quad (8)$$

where the equality stands up to some constant derivative factors.

Next we shall make a change of variables in the integral for the partition function given in Eq. (6) back to the original gauge field $A_\mu^a \rightarrow A_\mu^a - B_\mu^a$. Then we retrieve the original Lagrangian that does not depend on the gauge field plus an extra term gauge dependent according to:

$$\begin{aligned} \mathcal{L}_{B,A,c} & \rightarrow \mathcal{L}_{B=0,A} + i\theta(\mathcal{L}_{B=0,A,c})_1 + \mathcal{L}_B \\ \mathcal{L}_B & = -\bar{c}^a f^{abc} B_\mu^b \partial_\mu c^c - \bar{c}^a f^{amn} f^{nbc} B_\mu^m A_\mu^b c^c. \end{aligned} \quad (9)$$

Then Eqs. (6), (8) and (9) lead to the following master formula which will be used in everything that follows:

$$\exp\left[-\int d^4x \frac{1}{4g^2(\mu)} F_{\mu\nu}^a F^{\mu\nu} + i\theta(\mu) \frac{1}{32\pi^2} \int d^4x \tilde{F}^{a\mu\nu} F_{\mu\nu}^a + \text{higher order gauge invariants}\right] = \int dA_\mu^a d\bar{c}^b dc \exp\left[\int d^4x [\mathcal{L}_{B=0,A,c} + i\theta(\mathcal{L}_{B=0,A,c})_1 + \mathcal{L}_B]\right]. \quad (10)$$

In the end we shall make the notation:

$$\int d^4x [\mathcal{L}_{B=0,A,c} + i\theta(\mathcal{L}_{B=0,A,c})_1] = S(A, c, \theta). \quad (11)$$

III. THE PERTURBATIVE BETA FUNCTION

In this section we will set the theta term to zero and consider a background gauge field which satisfies the equation of motion. We apply the operators $\frac{\delta^2}{\delta B_\rho^m(z) \delta B_\rho^m(z)}$ and $\frac{\delta^4}{\delta B_\rho^m(z) \delta B_\rho^m(z) \delta B_\sigma^{m_1}(z) \delta B_\sigma^{m_1}(z)}$ to the left and right hand side of the Eq. (11). First we shall consider the right hand side:

$$\begin{aligned} W_1 &= \frac{\delta^2}{\delta B_\rho^m(z) \delta B_\rho^m(z)} \int dA_\mu^a d\bar{c}^b dc^b \exp\left[\int d^4x [\mathcal{L}_{B=0,A,c} + i\theta(\mathcal{L}_{B=0,A,c})_1 + \mathcal{L}_B]\right] = \\ &= \int dA_\mu^a d\bar{c}^b dc^b [\bar{c}^a(z) f^{amc} \partial_\rho c^c(z) + \bar{c}^a f^{amn} f^{nbc} A_\rho^b c^c] [\bar{c}^{a_1}(z) f^{a_1 m c_1} \partial^\rho c^{c_1}(z) + \bar{c}^{a_1} f^{a_1 m n_1} f^{n_1 b_1 c_1} A^{b_1 \rho} c^{c_1}] \times \\ &\exp\left[\int d^4x [\mathcal{L}_{B=0,A,c} + i\theta(\mathcal{L}_{B=0,A,c})_1 + \mathcal{L}_B]\right]. \end{aligned} \quad (12)$$

Since in the end we shall consider the value of this differential operator for $B = 0$ we observe (for the simplicity of the relations we shall omit all the indices knowing that the summation over them will lead to dimensionless coefficients whose specific values are irrelevant here):

$$W_1 = \text{const} \int dA_\mu^a d\bar{c}^b dc^b \bar{c}(z) \bar{c}(z) \frac{\delta}{\delta(\partial_\mu \bar{c}(z))} \frac{\delta}{\delta(\partial^\mu \bar{c}(z))} \exp\left[\int d^4x [\mathcal{L}_{B=0,A,c} + i\theta(\mathcal{L}_{B=0,A,c})_1 + \mathcal{L}_B]\right] \Big|_{B=0}. \quad (13)$$

where color indices of the ghost fields in the product are different. This is evident because the ghost term that contains B is of the form $-\bar{c} B^\mu D_\mu c$ whereas that that appears in the free action is just $\partial_\mu(\bar{c}) D_\mu c$.

Next we consider:

$$\begin{aligned} W_2 &= \frac{\delta^4}{\delta B_\rho^m(z) \delta B_\rho^m(z) \delta B_\sigma^{m_1}(z) \delta B_\sigma^{m_1}(z)} \int dA_\mu^a d\bar{c}^b dc \exp\left[\int d^4x [\mathcal{L}_{B=0,A,c} + i\theta(\mathcal{L}_{B=0,A,c})_1 + \mathcal{L}_B]\right] \Big|_{B=0} = \\ &= \int dA_\mu^a d\bar{c}^b dc^b \times \\ &\left[\bar{c}^a(z) f^{amc} \partial_\rho c^c(z) + \bar{c}^a f^{amn} f^{nbc} A_\rho^b c^c \right] [\bar{c}^{a_1}(z) f^{a_1 m c_1} \partial^\rho c^{c_1}(z) + \bar{c}^{a_1} f^{a_1 m n_1} f^{n_1 b_1 c_1} A^{b_1 \rho} c^{c_1}] \times \\ &[\bar{c}^{a_2}(z) f^{a_2 m_1 c_2} \partial_\sigma c^{c_2}(z) + \bar{c}^{a_2} f^{a_2 m_1 n_2} f^{n_2 b_2 c_2} A_\sigma^{b_2} c^{c_2}] [\bar{c}^{a_3}(z) f^{a_3 m_1 c_3} \partial^\sigma c^{c_3}(z) + \bar{c}^{a_3} f^{a_3 m_1 n_3} f^{n_3 b_3 c_3} A^{b_3 \sigma} c^{c_3}] \\ &\exp\left[\int d^4x [\mathcal{L}_{B=0,A,c} + i\theta(\mathcal{L}_{B=0,A,c})_1 + \mathcal{L}_B]\right] \Big|_{B=0}. \end{aligned} \quad (14)$$

Based on the same arguments as before one can write:

$$\begin{aligned} W_2 &= \text{const}' \int dA_\mu^a d\bar{c}^b dc^b \bar{c}(z) \bar{c}(z) \bar{c}(z) \bar{c}(z) \times \\ &\frac{\delta}{\delta(\partial_\mu \bar{c}(z))} \frac{\delta}{\delta(\partial^\mu \bar{c}(z))} \frac{\delta}{\delta(\partial_\rho \bar{c}(z))} \frac{\delta}{\delta(\partial^\rho \bar{c}(z))} \exp\left[\int d^4x [\mathcal{L}_{B=0,A,c} + i\theta(\mathcal{L}_{B=0,A,c})_1 + \mathcal{L}_B]\right] = \\ &\text{const}'' \int dA_\mu^a d\bar{c}^b dc^b \frac{\delta}{\delta(\partial_\mu \bar{c}(z))} \frac{\delta}{\delta(\partial^\mu \bar{c}(z))} \times \end{aligned}$$

$$\begin{aligned}
& \left[\bar{c}(z)\bar{c}(z)\bar{c}(z)\bar{c}(z)\frac{\delta}{\delta(\partial_\rho\bar{c}(z))}\frac{\delta}{\delta(\partial^\rho\bar{c}(z))}\exp\left[\int d^4x[\mathcal{L}_{B=0,A,c}+i\theta(\mathcal{L}_{B=0,A,c})_1+\mathcal{L}_B]\right] - \right. \\
& \text{const}'' \int dA_\mu^a d\bar{c}^b dc^b \left[\frac{\delta}{\delta(\partial_\mu\bar{c}(z))}\frac{\delta}{\delta(\partial^\mu\bar{c}(z))}\bar{c}(z)\bar{c}(z) \right] \times \\
& \bar{c}(z)\bar{c}(z)\frac{\delta}{\delta(\partial_\rho\bar{c}(z))}\frac{\delta}{\delta(\partial^\rho\bar{c}(z))}\exp\left[\int d^4x[\mathcal{L}_{B=0,A,c}+i\theta(\mathcal{L}_{B=0,A,c})_1+\mathcal{L}_B]\right] = \\
& \text{const}'' \int dA_\mu^a d\bar{c}^b dc^b \left[\frac{\delta}{\delta(\partial_\mu\bar{c}(z))}\frac{\delta}{\delta(\partial^\mu\bar{c}(z))}\bar{c}(z)\bar{c}(z) \right] \times \\
& \bar{c}(z)\bar{c}(z)\frac{\delta}{\delta(\partial_\rho\bar{c}(z))}\frac{\delta}{\delta(\partial^\rho\bar{c}(z))}\exp\left[\int d^4x[\mathcal{L}_{B=0,A,c}+i\theta(\mathcal{L}_{B=0,A,c})_1+\mathcal{L}_B]\right], \tag{15}
\end{aligned}$$

where we used the principles of integration by parts that work as well for noncommutative variables. Next we need to determine the quantity:

$$\begin{aligned}
& \left[\frac{\delta}{\delta(\partial_\mu\bar{c}(z))}\frac{\delta}{\delta(\partial^\mu\bar{c}(z))}\bar{c}(z)\bar{c}(z) \right] = \\
& \frac{\delta}{\delta(\partial_\mu\bar{c}(z))}\frac{\delta}{\delta(\partial^\mu\bar{c}(z))}\left[\int dy\bar{c}(y)\bar{c}(y)\delta(y-z)\right] = \\
& -\frac{\delta}{\delta(\partial_\mu\bar{c}(z))}\frac{\delta}{\delta(\partial^\mu\bar{c}(z))}\left[\int dy\bar{c}(y)\bar{c}(y)(\partial_\rho)^2\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}\exp[ip(y-z)]\right] \propto \\
& -\frac{\delta}{\delta(\partial_\mu\bar{c}(z))}\frac{\delta}{\delta(\partial^\mu\bar{c}(z))}\left[\int dy[(\partial_\rho)^2\bar{c}(y)\bar{c}(y)+\bar{c}(y)(\partial_\rho)^2\bar{c}(y)+2\partial_\rho\bar{c}(y)\partial^\rho\bar{c}(y)]\right] \times \\
& \int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}\exp[ip(y-z)] = -4\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}. \tag{16}
\end{aligned}$$

Here we used:

$$\begin{aligned}
& \frac{\delta(\partial_\rho\bar{c}(y))}{\delta(\partial_\mu\bar{c})} = \delta_{\mu\rho}\delta(z-y) \\
& \frac{\delta(\partial_\rho\partial_\rho\bar{c}(y))}{\delta(\partial_\mu\bar{c})} = \partial_\rho\delta(z-y) \\
& \int dx f(x)\delta'(a-x) = f(a)' \tag{17}
\end{aligned}$$

where the last equation in Eq. (17) is a generic property of the delta functions. By combining Eq. (15) and Eq. (16) we obtain:

$$W_2 = xW_1 \times \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \tag{18}$$

where x is a dimensionless constant that depends on the group factors and constants.

Next we need to compute the same derivatives for the left hand side of the Eq. (10). Since in the end we will set the field B to zero the results of these derivatives will be:

$$\begin{aligned}
& \frac{\delta^2}{\delta B_\rho^m(z)\delta B^{m\rho}(z)}\exp[-\Gamma(B)]|_{B=0} = a_0 \frac{\delta^2\Gamma(B)}{\delta B_\rho^m(z)\delta B^{m\rho}(z)}|_{B=0} \\
& \frac{\delta^4}{\delta B_\rho^m(z)\delta B^{m\rho}(z)\delta B_\sigma^{m_1}(z)\delta B^{m_1\sigma}(z)}\exp[-\Gamma(B)]|_{B=0} = \\
& a_1 \frac{\delta^4\Gamma[B]}{\delta B_\rho^m(z)\delta B^{m\rho}(z)\delta B_\sigma^{m_1}(z)\delta B^{m_1\sigma}(z)} + \\
& a_2 \frac{\delta^2\Gamma[B]}{\delta B_\rho^m\delta B^{m\rho}} \frac{\delta^2\Gamma[B]}{\delta B_\sigma^{m_1}\delta B^{m_1\sigma}} + \\
& a_3 \frac{\delta^2\Gamma[B]}{\delta B_\rho^m\delta B^{m_1\sigma}} \frac{\delta^2\Gamma[B]}{\delta B_\sigma^{m_1}\delta B^{m\rho}} + \text{similar terms to the previous ones.} \tag{19}
\end{aligned}$$

Here the coefficients a_0 , a_1 , a_2 and a_3 are dimensionless and since we do not aim to compute these relation exactly the dependence on the space time and internal indices is irrelevant. However the dependence on the variable space time is crucial. Then,

$$\begin{aligned} & \frac{\delta^2 \Gamma(B)}{\delta B_\rho^m(z) \delta B^{m\rho}(z)}|_{B=0} \propto \\ & \frac{\delta^2}{\delta B_\rho^m(z) \delta B^{m\rho}(z)} \left[\int d^4x \int d^4y \frac{d^4k}{(2\pi)^4} \frac{1}{2g^2} B_\mu^a(x) \exp[-ikx] [-k^2 g^{\mu\nu} + k^\mu k^\nu] B_\nu^a(y) \exp[iky] \right] = \\ & \frac{1}{2g^2} \int d^4x d^4y \frac{d^4k}{(2\pi)^4} \delta(x-z) \delta(y-z) \delta_{\mu\nu} \exp[-ikx + iky] [-k^2 g^{\mu\nu} + k^\mu k^\nu] = \\ & \frac{b_0}{g^2} \int \frac{d^4k}{(2\pi)^4} k^2, \end{aligned} \quad (20)$$

where the summation over space time indices is understood. Also b_0 is a dimensionless numerical coefficient irrelevant for our purposes. Moreover,

$$\begin{aligned} & \frac{\delta^4 \Gamma[B]}{\delta B_\rho^m(z) \delta B^{m\rho}(z) \delta B_\sigma^{m_1}(z) \delta B^{m_1\sigma}(z)} = \\ & \frac{1}{g^2} \frac{\delta^4}{\delta B_\rho^m(z) \delta B^{m\rho}(z) \delta B_\sigma^{m_1}(z) \delta B^{m_1\sigma}(z)} \int d^4x f^{abc} f^{ade} B_\mu^b(x) b_\nu^c(x) B_\mu^d(x) B_\mu^e(x) = \\ & \frac{1}{g^2} b_1 \int d^4x \delta(x-z) \delta(x-z) \delta(x-z) \delta(x-z) = b_1 \frac{1}{g^2} \left[\int \frac{d^4p}{(2\pi)^4} \right]^3. \end{aligned} \quad (21)$$

Here the coefficient b_1 is dimensionless but its value is irrelevant for our purposes. The contributions of the terms with coefficients a_2 and a_3 in the quadrilinear derivative in Eq. (19) can be determined easily from the square of the results in Eq. (20) only with different dimensionless coefficients. Then one may write:

$$\begin{aligned} & \frac{\delta^4}{\delta B_\rho^m(z) \delta B^{m\rho}(z) \delta B_\sigma^{m_1}(z) \delta B^{m_1\sigma}(z)} \exp[-\Gamma(B)]|_{B=0} = \\ & b_1 \frac{1}{g^2} \left[\int \frac{d^4p}{(2\pi)^4} \right]^3 + b_2 \frac{1}{g^4} \left[\int \frac{d^4k}{(2\pi)^4} k^2 \right]^2, \end{aligned} \quad (22)$$

where again b_2 is a dimensionless coefficient.

Next using Eqs (18), (20), (21) and (22) we determine for the quadrilinear derivative of the Eq. (10):

$$a \frac{1}{g^2} \left[\int \frac{d^4p}{(2\pi)^4} \right]^3 + b \frac{1}{g^4} \left[\int \frac{d^4k}{(2\pi)^4} k^2 \right]^2 = -c \frac{1}{g^2} \left[\int \frac{d^4k}{(2\pi)^4} k^2 \right] \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \int \frac{d^4r}{(2\pi)^4}, \quad (23)$$

where a , b and c are dimensionless coefficients that depend on the group constants.

We shall consider the expression in Eq. (23) in the dimensional regularization scheme (see Appendix A for the explicit calculations) which yields:

$$a \frac{1}{g^2 \mu^{2\epsilon}} \left[\frac{x_1}{\epsilon} + x_0 + \dots \right] + b \frac{1}{g^4 \mu^{4\epsilon}} [y_0 + y_1 \epsilon + \dots] + c \frac{1}{g^2 \mu^{2\epsilon}} \left[\frac{z_1}{\epsilon} + z_0 + \dots \right] = 0. \quad (24)$$

The above relation should be regarded as an infinite expansion in $\frac{1}{\epsilon}$ and cannot be used to extract directly g^2 from it because it will lead to the limit $\infty \times 0$ which is undetermined. But it is very useful to determine the beta function from it. For that we apply the operator $\frac{d}{d(\ln(\mu^2))}$ to the whole equation. This leads to:

$$\begin{aligned} & -a \frac{1}{g^4 \mu^{2\epsilon}} \beta(g^2) \left[\frac{x_1}{\epsilon} + x_0 + \dots \right] - a \epsilon \frac{1}{g^2 \mu^{2\epsilon}} \left[\frac{x_1}{\epsilon} + x_0 + \dots \right] - \\ & 2b \frac{1}{g^6 \mu^{4\epsilon}} [y_0 + y_1 \epsilon + \dots] \beta(g^2) - \\ & c \frac{1}{g^4 \mu^{2\epsilon}} \beta(g^2) \left[\frac{z_1}{\epsilon} + z_0 + \dots \right] - c \epsilon \frac{1}{g^2 \mu^{2\epsilon}} \left[\frac{z_1}{\epsilon} + z_0 + \dots \right] = 0. \end{aligned} \quad (25)$$

Next since the beta function is finite we equate the coefficients of the constant term to get:

$$(ax_0 + cz_0)\frac{1}{g^4}\beta(g^2) + (ax_1 + cz_1)\frac{1}{g^2} + 2by_0\frac{1}{g^6}\beta(g^2) = 0, \quad (26)$$

which can be further expressed as (by renaming the coefficients):

$$\beta(g^2) = -\frac{\beta_0 g^4}{1 - \frac{\beta_1}{\beta_0} g^2}, \quad (27)$$

where β_0 and β_1 are the first two orders positive and renormalization scheme independent coefficients of the beta function.

Note that the expression in Eq. (27) is exactly the Pica Sannino function.

One last important remark is in order. The effective action contains higher order derivative terms that we neglected. Terms with dimension 6 will not contribute to our derivatives because of the dimensionality. However terms of dimension 8 in essence might contribute. These are terms of the type $F_{\mu\nu}^a F^{a\mu\nu} F_{\rho\sigma}^b F^{b\rho\sigma}$ or similar terms with other internal or space time indices structure. However these terms contribute through their derivative parts and higher order derivative terms in a Lagrangian can always be reduced through the equation of motion which is already a constraint of our method. Thus we conclude that these terms can be ignored and that our results is a correct all order result.

IV. THE NON-PERTURBATIVE BETA FUNCTION

In this section we will show how the procedure employed in the previous section can be adjusted easily for the case when the theta angle term is introduced in the left hand side and right hand side of the formula in Eq. (10) and when B_μ^a can be assimilated to an arbitrary instanton solution. For that B_μ^a must not only satisfy the equation of motion but also the condition:

$$\int d^4x \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a = n \quad (28)$$

where $F_{\mu\nu}^a$ is the background gauge field tensor and n is the winding number. We shall label the instanton solution corresponding to the quantum number n by $(B_\mu^a)_n$. Our previous master formula applies as well in our case apart from these constraints. Each solution will depend on the instanton scale ρ . First we apply the quadratic and quadrilinear derivatives as before to the left hand side and right hand side of the Eq. (10). Because of the particularity of the functional derivatives the θ term will not contribute to the left hand side derivatives of $\Gamma[B_n]$. However the left hand side we will contain an extra term given by the $\exp[-\Gamma[B_n]]$. One should obtain for both sides more complicated relation that depend on the instanton solution. In the end we shall take the limit $B_n \rightarrow B_{n0}$ where B_{n0} is the particular solution for which $F^{a\mu\nu} = \text{sgn}(n)\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a$ and also take the limit $\rho \rightarrow \infty$. To see how this works we write the corresponding instanton solution for $n = 1$ [17]:

$$\begin{aligned} A_\mu^a &= 2\eta_{a\mu\nu}(x - x_0)_\nu \frac{1}{(x - x_0)^2 + \rho^2} \\ F_{\mu\nu}^a &= -4\eta_{a\mu\nu} \frac{\rho^2}{((x - x_0)^2 + \rho^2)^2}. \end{aligned} \quad (29)$$

Here,

$$\eta_{a\mu\nu} = \begin{bmatrix} \epsilon_{a\mu\nu} & \mu, \nu = 1, 2, 3 \\ -\delta_{a\nu} & \mu = 4 \\ \delta_{a\mu} & \nu = 4 \\ 0 & \mu = \nu = 4 \end{bmatrix}. \quad (30)$$

It turns out that this solution leads to a constant when the first order gauge invariant is integrated in the action in the limit $\rho \rightarrow \infty$ but tends to zero in the higher order gauge invariant or when the background gauge field does not appear in the right combination. Moreover the instanton solution can be set to zero in the right hand side of the master formula in Eq. (10) because it never appears in the right combination such that to lead to condition (28) and in the limit $\rho \rightarrow \infty$ all contributions go to zero. Then all our previous computations in section III apply as well here with the exception of the exponential of the instanton action that will appear on the left hand side of Eq. (10).

Relations between the bilinear and quadrilinear derivatives hold as well (We will assume that arbitrary instanton solutions share this nice feature). Considering all these facts one can write for the final result for the case when the background gauge field is an instanton solution analogous to Eq. (29).

$$\left[a \frac{1}{g^2 \mu^{2\epsilon}} \left[\frac{x_1}{\epsilon} + x_0 + \dots \right] + b \frac{1}{g^4 \mu^{4\epsilon}} [y_0 + y_1 \epsilon + \dots] \right] \exp \left[-\frac{8\pi^2 n}{g^2} + i\theta n \right] + \frac{c}{g^2 \mu^{2\epsilon}} \left[\frac{z_1}{\epsilon} + z_0 + \dots \right] = 0. \quad (31)$$

Since the exponential is essentially finite (we take $n > 0$ for simplicity but one can calculate similarly for an antiinstanton solution) we can further write:

$$a \frac{1}{g^2 \mu^{2\epsilon}} \left[\frac{x_1}{\epsilon} + x_0 + \dots \right] + b \frac{1}{g^4 \mu^{4\epsilon}} [y_0 + y_1 \epsilon + \dots] = -\frac{c}{g^2 \mu^{2\epsilon}} \left[\frac{z_1}{\epsilon} + z_0 + \dots \right] \exp \left[\frac{8\pi^2 n}{g^2} - i\theta n \right] \quad (32)$$

We differentiate as before the Eq. (32) and consider only the constant term to obtain:

$$\begin{aligned} -a \frac{ax_0}{g^4} \beta(g^2) - \frac{ax_1}{g^2} - 2 \frac{by_0}{g^6} \beta(g^2) = \\ \left[\left[\frac{cz_0}{g^4} \beta(g^2) + \frac{cz_1}{g^2} \right] \exp \left[\frac{8\pi^2 n}{g^2} - i\theta n \right] - \right. \\ \left. \frac{cz_0}{g^2} \left[-\frac{8\pi^2 n}{g^4} \beta(g^2) - i\beta(\theta)n \right] \exp \left[\frac{8\pi^2 n}{g^2} - i\theta n \right] \right]. \end{aligned} \quad (33)$$

By extracting the real and imaginary parts in Eq. (33) we obtain two relations between the two beta functions from which we determine:

$$\begin{aligned} \beta(\theta) &= \frac{\tan(\theta n)}{n} \left[\frac{1}{g^2} \beta(g^2) + \frac{8\pi^2 n}{g^4} \beta(g^2) + \frac{z_1}{z_0} \right] \\ \beta(g^2) &= -g^4 \left[ax_1 + cz_1 \cos(\theta n) \exp \left[\frac{8\pi^2 n}{g^2} \right] + cz_1 \tan(\theta n) \sin(\theta n) \exp \left[\frac{8\pi^2 n}{g^2} \right] \right] \times \\ &\quad \left[2by_0 + ax_0 g^2 + cz_0 g^2 \cos(\theta n) \exp \left[\frac{8\pi^2 n}{g^2} \right] + cz_0 8\pi^2 n \cos(\theta n) \exp \left[\frac{8\pi^2 n}{g^2} \right] + \right. \\ &\quad \left. cz_0 g^2 \tan(\theta n) \sin(\theta n) \exp \left[\frac{8\pi^2 n}{g^2} \right] + cz_0 \tan(\theta n) \sin(\theta n) \exp \left[\frac{8\pi^2 n}{g^2} \right] \right]^{-1}. \end{aligned} \quad (34)$$

Eq. (34) is the final results for the non-perturbative beta function in the background of an instanton solution with the winding number n .

V. A SOLUTION TO THE STRONG CP PROBLEM

In this section we shall determine the behavior of the theta angle in the nonperturbative regime where the coupling constant is large. Then from Eq. (34) we determine:

$$\beta(g^2) \approx -g^2 \frac{ax_1 \cos(\theta n) + cz_1}{ax_0 \cos(\theta n) + cz_0}. \quad (35)$$

In first order we also get:

$$\beta(\theta) \approx \frac{\tan(\theta n)}{n} \left[\frac{ax_0 z_1}{z_0} - ax_1 \right] \frac{\cos(\theta n)}{ax_0 \cos(\theta n) + cz_0} \quad (36)$$

This further leads to:

$$-\frac{dg^2}{g^2} = d\theta \frac{n}{\sin(\theta n)} \frac{1}{\frac{ax_0 z_1}{z_0} - ax_1} [ax_1 \cos(\theta n) + cz_1] \quad (37)$$

We integrate Eq. (37) from the finite coupling g_1^2 and theta angle θ_1 to the values $g_2^2 \approx \infty$ and θ_2 . This yields:

$$\begin{aligned} -\ln(g_2^2) + \ln(g_1^2) = & \frac{1}{\frac{ax_0 z_1}{z_0} - ax_1} \left[ax_1 [\ln(\sin(\theta_2 n)) - \ln(\sin(\theta_1 n))] + cz_1 [\ln(\tan(\frac{\theta_2 n}{2})) - \ln(\tan(\frac{\theta_1 n}{2}))] \right] = \\ & \frac{1}{\frac{ax_0 z_1}{z_0} - ax_1} \left[(ax_1 + cz_1) [\ln(\sin(\frac{\theta_2 n}{2})) - \ln(\sin(\frac{\theta_1 n}{2}))] + (ax_1 - cz_1) [\ln(\cos(\frac{\theta_2 n}{2})) - \ln(\cos(\frac{\theta_1 n}{2}))] \right]. \end{aligned} \quad (38)$$

First we note from Eqs. (26) and (27) that:

$$\frac{ax_1 + cz_1}{ax_0 + cz_0} = -\frac{\beta_0^2}{\beta_1} < 0, \quad (39)$$

from which we deduce $\frac{a}{c} < 0$ (see also Appendix A). Second we determine:

$$\begin{aligned} \frac{ax_1 + cz_1}{\frac{ax_0 z_1}{z_0} - ax_1} &= \frac{x_1 + \frac{c}{a} z_1}{\frac{x_0 z_1}{z_0} - x_1} > 0 \\ \frac{ax_1 - cz_1}{\frac{ax_0 z_1}{z_0} - ax_1} &= \frac{x_1 - \frac{c}{a} z_1}{\frac{x_0 z_1}{z_0} - x_1} < 0, \end{aligned} \quad (40)$$

where we used the results for x_0, x_1, z_0, z_1 from Appendix A. Then from Eq (40) one concludes that in order to have $-\infty$ for the left hand side of the Eq. (38) the right hand side must have $\theta_2 = 0$ (we particularize here for $n = 1$ but this solution is universal). Note that $\theta_2 = \pi$ cannot be a solution because of the relative signs of the terms in Eq. (38).

Consequently the effective theta angle in the nonperturbative regime is approximately zero fact indicated also by the experiments [15]. Thus our beta functions provide a clear solution to the strong CP problem. This results is maintained for the full QCD case because adding fermions to our theory does not alter in any way the major steps that we took in our derivation.

VI. DISCUSSION AND CONCLUSIONS

It is little known about the QCD beta function for the coupling constant in the non-perturbative region and even less about the behavior of the theta angle. In this work we first computed the all order perturbative beta function for the Yang Mills theory in the background gauge field method to find out that it has exactly the form proposed by Pica and Sannino in [8] by analogy to the NSVZ [9] beta function. Based on the global properties of the partition function in the background of an instanton field with the winding number n we further determined the non-perturbative beta functions for both the coupling constant and the theta angle. For that we extrapolate known properties of the instanton solution with $n = 1$ to arbitrary instantons with winding numbers n . The approach can be applied as well to antiinstantons. We then solve the two beta functions in the non-perturbative regime to obtain that the effective theta angle in this region is approximately zero. Since the main results of our paper can be easily extended to incorporate fermions we conclude that they are also valid for full QCD and thus constitute a solution to the strong CP problem.

Our work relates well with similar approximate results obtained in the literature for the non-perturbative beta functions [13], [14] and has some common features to the $N = 2$ supersymmetric beta functions calculated by Seiberg in [12].

Appendix A

Here we shall calculate explicitly the integrals in Eq. (23) in the dimensional regularization approach. We denote:

$$I_1 = \int \frac{d^d p}{(2\pi)^d}$$

$$\begin{aligned}
I_2 &= \int \frac{d^d p}{(2\pi)^d} p^2 \\
I_3 &= \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2}.
\end{aligned} \tag{A1}$$

In the standard dimensional regularization approach these integrals are considered zero [18]. However it is necessary to put these integrals in the context to see how one can reach such a result. This will show that in our method the contributions of these integrals are by far nontrivial. The integrals can be solved such that to have an ultraviolet cut-off or an infrared regulator. We will opt for the latter approach and introduce a small infrared regulator m^2 . We use the master formula [18] in the euclidean space,

$$\begin{aligned}
&\int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^\alpha}{(p^2 + m^2)^\beta} = \\
&\frac{\pi^d}{(2\pi)^d} m^{d+2\alpha-2\beta} \frac{\Gamma[\alpha + \frac{d}{2}] \Gamma[\beta - \alpha - \frac{d}{2}]}{\Gamma[\frac{d}{2}] \Gamma[\beta]},
\end{aligned} \tag{A2}$$

and further write for the integrals in Eq. (A1):

$$\begin{aligned}
I_1 &= \int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^\alpha}{(p^2 - m^2)^\alpha} \\
I_2 &= \int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^{\alpha+1}}{(p^2 - m^2)^\alpha},
\end{aligned} \tag{A3}$$

where $\alpha = 0$ and further,

$$I_3 = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2}. \tag{A4}$$

We need to calculate:

$$\begin{aligned}
X_1 &= I_1^3 \\
X_2 &= I_2^2 \\
I_3 &= I_1 I_2 I_3.
\end{aligned} \tag{A5}$$

We shall consider m^2 small and finite and only in the end take the limit $m \rightarrow 0$. But then one can divide Eq. (23) safely to I_2^2 because the equality will be respected. In the end the quantities of interest will be:

$$\begin{aligned}
Y_1 &= \frac{I_1^3}{I_2^2} \\
Y_2 &= \frac{I_2^2}{I_2^2} = 1 \\
Y_3 &= \frac{I_1 I_2 I_3}{I_2^2} = \frac{I_1 I_3}{I_2}.
\end{aligned} \tag{A6}$$

In the dimensional regularization (\overline{MS}) with $d = 4 - 2\epsilon$ we get:

$$\begin{aligned}
Y_1 &= \frac{9}{128\pi^2\epsilon} + \frac{33}{256\pi^2} + \dots \\
Y_3 &= \frac{3}{32\pi^2\epsilon} + \frac{7}{64\pi^2} + \dots
\end{aligned} \tag{A7}$$

Thus one can extract:

$$\begin{aligned}
x_1 &= \frac{9}{128\pi^2} \\
x_0 &= \frac{33}{256\pi^2} \\
y_1 &= \frac{3}{32\pi^2} \\
y_0 &= \frac{7}{64\pi^2}.
\end{aligned} \tag{A8}$$

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